Three loop renormalization group for a marginally perturbed SU(2) WZW model

Chigak Itoi *

Department of Physics,

College of Science and Technology, Nihon University,

Kanda Surugadai, Chiyoda, Tokyo 101, Japan

Abstract

Employing a simple calculation method obtained by M.-H. Kato, we calculate the three loop renormalization group in the $su(2)$ coset conformal field theory with a slightly relevant perturbation and the $su(2)$ Wess-Zumino-Witten model with a particular invariant marginal perturbation. Zamolodchikov's c-theorem, exact data of the perturbation operator and a known exact form of the operator product coefficient enable us to calculate the beta function, the gamma function and the c-function to three loop order. This result gives the logarithmic finite size correction to the ground state energy and the low temperature behavior of the specific heat in the Heisenberg antiferromagnetic chain with high accuracy. We describe the consistency with results obtained by several authors on the basis of its exact solvability. We discuss an experiment of the specific heat and the suceptibility recently observed.

PACS numbers: 75.10.Jm, 11.10.Hi, 11.25.Hf

keywords: c-function, renormalization group, marginal operator, three loop, logarithmic correction

*e-mail: itoi@phys.cst.nihon-u.ac.jp

1 Introduction

There have been extensive studies in conformal field theory (CFT) with a relevant perturbation which reveal universal nature of critical phenomena in low dimensional statistical physics. Landau-Ginzburg theory gives a good qualitative picture for the renormalization group (RG) flow including multi-critical points [1]. In this theory, we can specify the driving relevant operator of the CFT at the ultraviolet (UV) fixed point and the irrelevant operator of another CFT at the infrared (IR) fixed point. We can check the expected flow practically in the ε-expansion by calculating the deviation of the central charge which obeys Zamolodchikov's c-theorem [2].

In this paper, we utilize Zamolodchikov's c-theorem and the exact form of the operator product expansion (OPE) coefficient to calculate the beta function to three loop order in the ε-expansion for $su(2)$ coset models with the slightly relevant perturbation with the dimension $2-\varepsilon$. M. -H. Kato argued this application of the c-theorem to calculate the beta function up to two loop order for the slightly perturbed general coset models in all A-D-E classes [3]. By taking the limit $\varepsilon \to 0$, the coset CFT becomes a certain Wess-Zumino-Witten (WZW) model with a particular invariant marginal perturbation. In this limit, he also calculated the logarithmic finite size correction to the ground state energy and the logarithmic temperature dependence of the specific heat of quantum spin chains to two loop order. The obtained fitting function agrees with the groud state energy of an $su(2)$ spin chain by a numerical Bethe ansatz [4]. Here, we calculate higher order logarithmic correction to the ground state energy and discuss the consistency with results obtained by several authors in different methods. This paper is organized as follows. In section 2, we review M.-H. Kato's method to calculate two loop beta function in general coset models with generalizing some equations for higher order calculation. In section 3, we confine our discussions to $su(2)$ case. In $su(2)$ coset model, the known exact form of the OPE coefficient of the driving operator enables us to calculate the three loop beta function. In section 4, we discuss the logarithmic finite size correction to low lying energy levels. We discuss consistency with a numerical Bethe ansatz and some other methods. In section 5, we calculate a logarithmic temperature dependence of the specific heat per unit length in a $su(2)$ spin chain. We discuss the recent experiment [5] from the view point of the higher order renormalization group calculation.

2 Two loop calculation in general coset models

Here, we review M.-H. Kato's method for two loop beta function [3] with preparing some extended forms of the equations for three loop calculation in the next section. In this method, we consider perturbed coset CFT $\mathcal{M}(k,l;G) = \hat{G}_k \oplus \hat{G}_l/\hat{G}_{k+l}$ by a slightly relevant operator Φ_{UV} with self-closing algebra which corresponds to the (1,3) operator in the Virasoro minimal model as a regularized theory for the marginally perturbed WZW model. In this section, we assume those following two facts obtained by Ahn, Bernard and Leclair [7]. They showed that the slightly relevant operator Φ_{UV} drives the CFT at UV fixed point to the IR one with the

irrelevant perturbation of the operator $\Phi_{\rm IR}$, as in $su(2)$ case [1, 2]

$$H(k,l)+g\int\frac{d^2z}{2\pi}\Phi_{\rm UV}(z,\bar{z})\to H(k,l-k)+g'\int\frac{d^2z}{2\pi}\Phi_{\rm IR}(z\bar{z}),\ \text{(IR limit)}\qquad(1)$$

where $H(k,l)$ is a critical hamiltonian in $\mathcal{M}(k,l;G)$. They showed also that this deformed CFT $\mathcal{M}(k,l;G)$ by the operator $\Phi_{\rm UV}$ becomes the $\hat{G}_k$ WZW model with an invariant marginal perturbation, that is

$$\hat{G}_k\oplus\hat{G}_l/\hat{G}_{k+l}\to\hat{G}_k,\ (l\to\infty),$$

and

$$\Phi_{\rm UV}(z,\bar{z})\to\Phi(z,\bar{z})\equiv-\frac{2}{k\sqrt{D}}\sum_{a=1}^{D}J^a(z)\bar{J}^a(\bar{z}),\ (l\to\infty),\qquad(2)$$

in a limit $l\to\infty$, where D is the dimension of the Lie algebra G. The existence of RG flow between these fixed points is guaranteed by the self-closing algebra of the operator $\Phi_{\rm UV}$. In the CFT $\mathcal{M}(k,l;G)$, the central charge is

$$\begin{aligned}c(k,l;G)&=c(\hat{G}_k)+c(\hat{G}_l)-c(\hat{G}_{k+l})\\&=\frac{kD}{k+h_c}\left(1-\frac{h_c(k+h_c)}{(l+h_c)(k+l+h_c)}\right),\end{aligned}\qquad(3)$$

and the conformal dimensions of the operators $\Phi_{\rm UV}$ and $\Phi_{\rm IR}$ are

$$\begin{aligned}\Delta_{\rm UV}&=1-\frac{h_c}{k+l+h_c}\\\Delta_{\rm IR}&=1+\frac{h_c}{l-k+h_c},\end{aligned}\qquad(4)$$

where r is the rank and h_c is the dual coxeter number of the Lie algebra G. The dimension of the Lie algebra is given by $D=r(1+h_c)$ in simply-laced algebra. For example in $G=su(n)$, these are given by $r=n-1$, $h_c=n$ and $D=n^2-1$. Here, we define the parameter

$$\varepsilon\equiv2-2\Delta_{\rm UV}=\frac{2h_c}{k+l+h_c},\qquad(5)$$

and discuss the RG flow from the UV theory $\mathcal{M}(k,l;G)$ to the IR theory $\mathcal{M}(k,l-k;G)$ in the ε-expansion. The limit $\varepsilon\searrow0$ as $l\nearrow\infty$ gives

$$\Delta_{\rm UV}=1-\frac{\varepsilon}{2}\to1.$$

This relation is necessary for the identification of the operator (2), which should be shown by the correspondence between these OPE coefficients with their closing algebra, as well as their conformal dimensions. The OPE relations of the operator $\Phi_{\rm UV}$ and the marginal operator are

$$\Phi_{\rm UV}(z,\bar{z})\Phi_{\rm UV}(0,0)\sim\frac{b(\varepsilon)}{|z|^{2\Delta_{\rm UV}}}\Phi_{\rm UV}(0,0)\qquad(6)$$

$$\Phi(z,\bar{z})\Phi(0,0)\sim\frac{b_0}{|z|^2}\Phi(0,0).\qquad(7)$$

with $b(\varepsilon) \to b_0$ as $\varepsilon \to 0$. To obtain the OPE coefficient $b(\varepsilon)$ in the ε-expansion, we assume Zamolodchikov's c-theorem [2]. The beta function of the deformed CFT by the operator $\Phi_{\rm UV}$ with the running coupling constant g is

$$\beta(g) = -\varepsilon g + \frac{b(\varepsilon)}{2} g^2 + \frac{d(\varepsilon)}{2} g^3 + \frac{e(\varepsilon)}{2} g^4 + \cdots, \tag{8}$$

and the gamma function of the operator $\Phi_{\rm UV}$ is given by

$$\gamma(g) = 2 + \frac{\partial \beta(g)}{\partial g}. \tag{9}$$

The coefficients in the beta function is expanded in ε

$$\begin{aligned} b(\varepsilon) &= \sum_{n=0}^{\infty} b_n \varepsilon^n, & (10)\\ d(\varepsilon) &= \sum_{n=0}^{\infty} d_n \varepsilon^n, & (11)\\ e(\varepsilon) &= \sum_{n=0}^{\infty} e_n \varepsilon^n, & (12)\\ &\cdots. & (13) \end{aligned}$$

The beta function (8) has a trivial fixed point $g = 0$ and another non-trivial one $g = g_* \neq 0$. The trivial fixed point $g = 0$ is the UV CFT $\mathcal{M}(k, l; G)$ and the other one g_* corresponds to the IR CFT $\mathcal{M}(k, l-k; G)$. The non-trivial fixed point is expanded in a series

$$g_* = \sum_{n=1}^{\infty} g_n \varepsilon^n, \tag{14}$$

whose coefficients g_n are written in terms of those of the beta function (13)

$$\begin{aligned} g_1 &= \frac{2}{b_0}, & (15)\\ g_2 &= -\frac{1}{b_0^3}\left(2 b_1 b_0^2 + 4 d_0\right), \\ g_3 &= \frac{2}{b_0^5}\left(b_0^2 b_1^2 - b_0^3 b_2 + 6 b_0 b_1 d_0 + 8 d_0^2 - 2 b_0^2 d_1 - 4 b_0 e_0\right), \\ &\cdots. & (16) \end{aligned}$$

Zamolodchikov's c-function is defined by

$$c(g) = c_{\rm UV} + \frac{3}{2}\int_0^g \beta(x) dx, \tag{17}$$

where $c_{\rm UV}$ is the central charge of the CFT $\mathcal{M}(k, l; G)$. Zamolodchikov's c-theorem gives us the following constraints

$$\begin{aligned} c_{\rm IR} &= c(g_*), & (18)\\ \Delta_{\rm IR} &= \gamma(g_*)/2 = 1 + \beta'(g_*)/2, & (19) \end{aligned}$$

where $c_{\rm IR}$ is the central charge of the IR CFT $\mathcal{M}(k, l-k; G)$. The right hand side in these constraints can be expanded in ε

$$c_{\rm IR} - c_{\rm UV} = -\frac{\varepsilon^3}{b_0^2} + \left(\frac{2b_1}{b_0^3} + \frac{3d_0}{b_0^4}\right)\varepsilon^4 + \left(\frac{2b_2}{b_0^3} + \frac{3d_1 - 3b_1^2}{b_0^4} + \frac{-60b_1d_0 + 24e_0}{5b_0^5} - \frac{12d_0}{b_0^6}\right)\varepsilon^5 + \cdots, \tag{20}$$

and

$$\Delta_{\rm IR} = 1 + \frac{\varepsilon}{2} + \frac{d_0}{b_0^2}\varepsilon^2 + \frac{1}{b_0^4}\left(-2b_0b_1d_0 - 4d_0^2 + b_0^2d_1 + 4b_0e_0\right)\varepsilon^3 + \cdots. \tag{21}$$

The knowledge of the flow $\mathcal{M}(k, l; G) \to \mathcal{M}(k, l-k; G)$ by the slightly relevant perturbation gives further constraints

$$c_{\rm IR} = c(k, l-k; G) = \frac{kD}{k+h_c}\left(1 - \frac{h_c(k+h_c)}{(l-k+h_c)(l+h_c)}\right), \tag{22}$$

$$\Delta_{\rm IR} = 1 + \frac{h_c}{l-k+h_c}. \tag{23}$$

The ε-expansion of these constraints (22) and (23) can be done as follows:

$$c_{\rm IR} - c_{\rm UV} = -\frac{k^2D\varepsilon^3}{4h_c^2}\left(1 + \frac{3k}{2h_c}\varepsilon + \frac{7k^2}{4h_c^2}\varepsilon^2 + \cdots\right), \tag{24}$$

$$\Delta_{\rm IR} = 1 + \frac{\varepsilon}{2} + \frac{k}{2h_c}\varepsilon^2 + \frac{k^2}{2h_c^2}\varepsilon^3 + \cdots. \tag{25}$$

These constraints (20), (21), (24) and (25) can be solved with respect to b_1 and d_0 in terms of b_0

$$b_1 = -\frac{3k}{2h_c}b_0, \qquad d_0 = \frac{k}{2h_c}b_0^2, \tag{26}$$

where b_0 is OPE coefficient of the marginal operator in the WZW model

$$b_0 = \frac{2h_c}{k\sqrt{D}}.$$

Therefore, we obtain the beta function of the marginally perturbed $\hat{G}_k$ WZW model up to two loop order

$$\beta(g) = \frac{h_c}{k\sqrt{D}}g^2 + \frac{h_c}{kD}g^3,$$

without explicit calculation.

3 Three loop beta function

In the previous section, we obtained the two loop beta function with the help of two assumptions (1) and (2). In this section, we limit our discussion to $G = su(2)$ case and we check these assumption within the framework of the ε-expansion. In this case, we have $D = 3$, $h_c = 2$ and $r = 1$. We have the slightly relevant operator $\Phi_{\rm UV}$

with the conformal dimension $\Delta_{\rm UV} = 1 - 2/(k+l+2)$. The exact form of its OPE coefficient $b(\varepsilon)$ in the CFT $\mathcal{M}(k,l;su(2))$ is given in [8]

$$\begin{aligned} b(\varepsilon) &= \frac{8}{\sqrt{3}k}\frac{(2-(k+1)\varepsilon)^2}{(2-\varepsilon)(4-(k+2)\varepsilon)} \\ &\times \frac{\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)}\left(\frac{\Gamma(1-\varepsilon/4)}{\Gamma(1+\varepsilon/4)}\right)^{3/2}\left(\frac{\Gamma(1+3\varepsilon/4)}{\Gamma(1-3\varepsilon/4)}\right)^{1/2}\left(\frac{\Gamma(1+\varepsilon/2)}{\Gamma(1-\varepsilon/2)}\right)^{2}. \end{aligned} \tag{27}$$

This gives us the data of $b_0,\ b_1,\ b_2, \cdots$ and then the method explained in the previous section enables us to calculate the beta function up to third order. The OPE coefficient (28) is expanded in

$$b(\varepsilon) = \frac{4}{\sqrt{3}k}\left(1 - \frac{3k}{4}\varepsilon + \frac{k(k-6)}{16}\varepsilon^2 + \cdots\right). \tag{28}$$

This formula shows

$$b(\varepsilon) \to b_0 \equiv \frac{4}{\sqrt{3}k} \quad (\varepsilon \to 0),$$

which justifies the identification (2) we assumed. The one loop beta function is given by $\beta(g) = b_0 g^2/2$ which enables us to calculate the central charge $c_{\rm IR}$ and the conformal dimension $\Delta_{\rm IR}$ by eq.(20) and eq.(21). These results show that the expected flow (1) is checked in the ε-expansion around the UV fixed point at one loop level. Therefore, we can use the constrains to solve the higher order coefficients in the ε-expansion. Since the constraints (20), (21), (24) and (25) can be solved with respect to d_0, d_1, and e_0 in terms of b_0, b_1, and b_2, we obtain them explicitly

$$d_0 = \frac{4}{3k}, \quad d_1 = -\frac{10(k-2)}{9k}, \quad e_0 = \frac{10(k-2)}{9\sqrt{3}k^2}, \tag{29}$$

where the rank and the dual coxeter number of $su(2)$ are given by $r = 1$ and $h_c = 2$. The limit $\varepsilon \to 0$ gives the beta function to three loop order for the marginally perturbed level k $su(2)$ WZW model

$$\beta(g) = \frac{b_0}{2}g^2 + \frac{d_0}{2}g^3 + \frac{e_0}{2}g^4 + {\rm O}(g^5). \tag{30}$$

4 The c-function and the logarithmic correction

Here, we consider the perturbed WZW model in the asymptotically non-free region $g > 0$. The marginally perturbed level k $su(2)$ WZW model in asymptotically non-free region describes $S = k/2$ antiferromagnetic Faddeev-Takhtajan model. Especially for $k = 1$, this corresponds to $S = 1/2$ Heisenberg antiferromagnetic chain model. We calculate the c-function by integrating the beta function and discuss the logarithmic finite size correction to the ground state energy of the antiferromagnetic chain. First, we integrate the differential equation

$$\frac{dg}{d\log l} = -\beta(g) \tag{31}$$

for the running coupling constant $g(l)$

$$I(g(l)) - I(g(a)) = \log l/a, \tag{32}$$

where a is the UV cut off length scale of the field theory. Here, we define the integral function $I(x)$ by

$$I(x) \equiv -\int \frac{dx}{\beta(x)} = \frac{2}{b_0 x} + \frac{2d_0}{b_0^2}\log x - \frac{2}{b_0^2}\left(\frac{d_0^2}{b_0} - e_0\right)x + \mathrm{O}(x^2). \tag{33}$$

Let L be the size of the chain which sufficiently large compared to the UV cut off a. In this case, we have $g(L) << g(a)$ because of the asymptotic non-freedom. Here, we introduce the RG invariant length scale $L_0 \equiv a\exp(-I(g(a)))$. The running coupling constant $g(L)$ is determined by

$$I(g(L)) = \log L/L_0,$$

and solve this equation by expanding $1/I(g(L))$ in $g(L)$ and $s \equiv 1/\log(L/L_0)$

$$g(L) = \frac{\sqrt{3}k}{2}s\left(1 + s\frac{k}{2}\log s + s^2\left(\frac{k^2}{4}\log^2 s - \frac{k(k+10)}{24}\right) + \cdots\right). \tag{34}$$

Substituting this running coupling constant into the c-function (17), we obtain

$$\begin{aligned} c(g(L)) &= c_0 + \frac{A_1}{\log^3 L/L_0} + \frac{A_2 + A_2'\log\log L/L_0}{\log^4 L/L_0} \\ &+ \frac{A_3 + A_3'\log\log L/L_0 + A_3''(\log\log L/L_0)^2}{\log^5 L/L_0} + \mathrm{O}(\log^{-6} L/L_0), \end{aligned} \tag{35}$$

where the central charge c_0 of level k $su(2)$ WZW model and other coefficients are

$$\begin{aligned} c_0 &= \frac{3k}{(k+2)}, \\ A_1 &= \frac{3}{8}k^2, \\ A_2 &= \frac{9}{64}k^3, \quad A_2' = -\frac{9}{16}k^3, \\ A_3 &= -\frac{9}{16}k^3, \quad A_3' = -\frac{9}{32}k^4, \quad A_3'' = \frac{9}{16}k^4. \end{aligned} \tag{36}$$

This formula has only one fitting parameter L_0. The logarithmic finite size correction to the ground state energy $E_0(L)$ in the Faddeev-Takhtajan chain with finite length L is calculated by

$$E_0(L) = e_\infty L - \frac{\pi\hbar v}{6L}c(g(L)) \tag{37}$$

with the formula (36), where e_∞ is energy density in the infinite length limit and v is the spin wave velocity. This formula (36) fits well the data of the ground state energy in a numerical Bethe ansatz obtained by Nomura [4] in $k = 1$ case. He shows the row data of low lying energy levels in the Heisenberg antiferromagnetic chain with $L = 256$ - 16384 lattice sites with unit lattice spacing. To check the consistency

with all data including the excited energy, we have to consider the gamma functions of two primary fields in WZW model. In the level one $su(2)$ WZW model, we have four primary fields which correspond a singlet and a triplet excitations with conformal dimension (1/4, 1/4). We write their gamma function

$$\begin{aligned} \gamma^t(g) &= \gamma_0^t + \gamma_1^t b_0 g + \gamma_2^t (b_0 g)^2 + \gamma_3^t (b_0 g)^3 + \mathrm{O}(g^4), \\ \gamma^s(g) &= \gamma_0^s + \gamma_1^s b_0 g + \gamma_2^s (b_0 g)^2 + \gamma_3^s (b_0 g)^3 + \mathrm{O}(g^4). \end{aligned} \tag{38}$$

The level one $su(2)$ WZW model gives

$$\gamma_0^t = \gamma_0^s = 1/2,$$

and one loop RG calculation gives

$$\gamma_1^t = -1/2, \quad \gamma_1^s = 3/2, \quad b_0 = \frac{4}{\sqrt{3}},$$

however other coefficients have never been calculated. Note that the first excited state has three fold degeneracy because of the $su(2)$ symmetry. First, we fit the data at two loop level. In this order, the invariant scale is given by $L_0^{-1} = 1.74087$ for fitting the data of the ground state energy in [4] by the formula (36). Unfortunately in the present stage, we have only the first order gamma function of the primary fields. Though we cannot check his data of excited energy in second order explicitly, we can fix the coefficients γ_2^t and γ_2^s by fitting the data as follows:

$$\begin{aligned} \gamma_{2Loop}^t(g) &= 1.00012 \left(\frac{1}{2} - \frac{1}{2} b_0 g \right) - 0.0619577 (b_0 g)^2, \\ \gamma_{2Loop}^s(g) &= 0.999499 \left(\frac{1}{2} + \frac{3}{2} b_0 g \right) + 1.00147 (b_0 g)^2. \end{aligned} \tag{39}$$

Next, we discuss three loop order. The invariant length scale becomes $L_0^{-1} = 2.14538$ to fit the ground state energy in this order. The gamma functions for fitting the excited energy become

$$\begin{aligned} \gamma_{3Loop}^t(g) &= 0.999868 \left(\frac{1}{2} - \frac{1}{2} b_0 g \right) - 0.00772577 (b_0 g)^2 - 0.284163 (b_0 g)^3, \\ \gamma_{3Loop}^s(g) &= 0.999932 \left(\frac{1}{2} + \frac{3}{2} b_0 g \right) + 0.863347 (b_0 g)^2 + 0.64291 (b_0 g)^3. \end{aligned} \tag{40}$$

It is important that both coefficients of the first order are quite close to unity as in two loop order. This fact shows a good agreement between our three loop calculation and the numerical Bethe ansatz. On the other hand, other coefficients change from those in the two loop order.

In this fitting, we compare our calculation of some typical values with those by Nomura in the numerical Bethe ansatz

$$\begin{aligned} c_{Bethe}(g(256)) &= 1.00103233, \quad c_{Bethe}(g(16384)) = 1.000239164, \\ c_{2Loop}(g(256)) &= 1.00101926, \quad c_{2Loop}(g(16384)) = 1.000241815, \\ c_{3Loop}(g(256)) &= 1.00101115, \quad c_{3Loop}(g(16384)) = 1.000243613, \end{aligned} \tag{41}$$

$$
\begin{aligned}
\gamma^t_{Bethe}(g(256)) &= 0.464830, \quad \gamma^t_{Bethe}(g(16384)) = 0.478297, \\
\gamma^t_{2Loop}(g(256)) &= 0.464835, \quad \gamma^t_{2Loop}(g(16384)) = 0.478301, \\
\gamma^t_{3Loop}(g(256)) &= 0.464832, \quad \gamma^t_{3Loop}(g(16384)) = 0.478296,
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\gamma^s_{Bethe}(g(256)) &= 0.620490, \quad \gamma^s_{Bethe}(g(16384)) = 0.570912, \\
\gamma^s_{2Loop}(g(256)) &= 0.620513, \quad \gamma^s_{2Loop}(g(16384)) = 0.570916, \\
\gamma^s_{3Loop}(g(256)) &= 0.620493, \quad \gamma^s_{3Loop}(g(16384)) = 0.570910,
\end{aligned} \tag{43}
$$

The two loop calculation is slightly better than the three loop one with respect to the c-function. It is not less trivial that this three loop gamma functions fit the excited energy levels very well, since this fitting has two more parameters γ^t_3, γ^s_3. These results of the RG calculation seem consistent with the numerical Bethe ansatz.

Here, we discuss consistency of our results with those obtained by several authors [4, 9, 10, 11]. First, we comment on Nomura's fitting of his numerical data based on a two loop RG calculation [4], which gives $A_1 = 0.365162$. He take into account only first order of corrections to the c-function, although he uses the running coupling constant obtained from the two loop beta function. This is consistent way to take into account $\log\log L/L_0 \log^{-4} L/L_0$ term in the c-function (36). However, we cannot neglect $\log^{-4} L/L_0$ term to fit the numerical data, since $\log^{-4} L/L_0$ term behaves as large as $\log\log L/L_0 \log^{-4} L/L_0$ in numerical data of the model with not so large degrees of freedom (256-16384 sites). To obtain more accurate fitting, we should take into account g^4 order in c-function and g^2 order in the gamma function of the primary fields. He fixes the parameter $L_0^{-1} = 0.56532$ in such a way that the one loop correction to the triplet excitation energy with the two loop running coupling constant fits the numerical data of the triplet excitation. It is possible to fit only one observable by adjusting the parameter L_0 with neglecting the second order correction in it, since the second order logarithmic correction can be absorbed into the first order one by adjusting the parameter L_0. In this procedure however, the second order logarithmic correction cannot be absorbed into first order one to fit observables more than one without the second order correction. Despite his recognition of the g^2 order correction to the singlet excitation energy and L^{-2} correction to the ground state energy by the irrelevant operator $L_{-2}\bar{L}_{-2}\mathbf{1}$, his obtained c-function is $c(g(L)) = 1 + 0.36516\left(\frac{b_0 g}{2}\right)^3 + \frac{1.666}{L^2}$, which gives not so good agreement with the leading coefficient $A_1 = 0.375$. If we take care of $\log^{-4} L/L_0$ term from the g^4 term in the c-function and $\log^{-2} L/L_0$ term from the g^2 term in the gamma function $\gamma^t(g)$, the numerical data should be fitted with sufficient accuracy even in two loop order. Actually, Karbach and Mütter obtain $A_1 = 0.375$ $A'_2 = -0.73$ and $A_2 = 0.15$ with recognition of $\log^{-4} L/L_0$ term in the c-function with a careful treatment of fitting their data in numerical Bethe ansatz. Their obtained A_1 agrees with ours, and also Klümper shows this result $A_1 = 0.375$ by calculating the specific heat in his new method of solving a set of non-linear equations based on a quantum transfer matrix for the thermodymamic quantities partially using a numerical calculation. We will see the relation between the specific heat and the ground state energy in the next section. Next, we discuss the two loop order coefficients. Our results $A'_2 = -9/16 = -0.5625$ and $A_2 = 9/64 + 9/8 \log L_0 \sim -0.483058$, with the two loop fitting parameter $L_0^{-1} = 1.74087$ in (36) does not agree with those by Karbach and

Mütter. We expect our formula with the three loop correction (36) will be useful to fit their data, eventhough fitting the numerical data for the higher order coefficients seems extremly difficult.

Finally, we mention the results obtained by Woynarovich and Eckel with Euler-MacLaurin expansion in analytic Bethe ansatz [11] . The coefficients γ_0^t, γ_1^t, γ_0^s and γ_1^s in excited energy level and c_0 in the WZW model agree with those obtained by Euler-MacLaurin expansion in the Bethe ansatz. On the other hand, the first coefficient in the logarithmic correction in eq.(37) is calculated as $A_1 = 0.343347\cdots$, in [11] which does not agree with the result $A_1 = 3/8 = 0.375$ obtained by the WZW model. This is well-known contradiction pointed out in [4, 9, 10, 12]. Since our fitting of numerical Bethe ansatz, an appropriate treatment of the numerical Bethe ansatz [9] and Klümper's result [10] support the WZW model, we believe $A_1 = 3/8$.

5 Specific heat and suceptibility

Next, we calculate the specific heat per unit length. The partition function of one dimensional quantum system is given by

$$Z(L, M) = \mathrm{tr}\, e^{-MH/\hbar v}. \tag{44}$$

$M = \frac{\hbar v}{k_{\mathrm{B}} T}$ has the dimension of length, where v is a spin wave velocity, k_{B} is the Boltzmann constant and T is the temperature. The partition function possesses a modular invariance

$$Z(L, M) = Z(M, L), \tag{45}$$

and the free energy per unit length $f(L, M) = -\frac{k_{\mathrm{B}} T}{L} \log Z(L, M)$ possesses it as well. In the low temperature limit $M \to \infty$, the free energy becomes

$$\begin{aligned} f(L, \infty) &= E_0(L)/L \\ &= e_\infty - \frac{\pi \hbar v}{6L^2} c(g(L)). \end{aligned} \tag{46}$$

The modular invariance gives the temperature dependence of the free energy per unit length for the infinitely long chain as follows:

$$\begin{aligned} f(\infty, M) &= f(M, \infty) \\ &= e_\infty - \frac{\pi \hbar v}{6M^2} c(g(M)). \end{aligned} \tag{47}$$

The specific heat per unit length $C(T) = -T \frac{\partial^2 f}{\partial T^2}$ is calculated in a low temperature expansion

$$\begin{aligned} C(T) &= \frac{\pi k_{\mathrm{B}}^2 T}{3\hbar v} (c_0 + \frac{B_1}{\log^3 T_0/T} + \frac{B_2 + B_2' \log\log T_0/T}{\log^4 T_0/T} \\ &+ \frac{B_3 + B_3' \log\log T_0/T + B_3''(\log\log T_0/T)^2}{\log^5 T_0/T} + \mathrm{O}(\log^{-6} T_0/T)\ ), \end{aligned} \tag{48}$$

where $T_0 = \frac{\hbar v}{k_{\rm B} L_0}$ is a parameter with the dimension of temperature. The coefficients are given by

$$\begin{aligned} c_0 &= \frac{3k}{k+2} \\ B_1 &= \frac{3}{8}k^2, \\ B_2 &= \frac{9}{64}k^2(k+4), \quad B_2' = -\frac{9}{16}k^3, \\ B_3 &= -\frac{9}{16}k^2(k-4), \quad B_3' = -\frac{9}{32}k^3(k+4), \quad B_3'' = \frac{9}{16}k^4. \end{aligned} \tag{49}$$

This formula shows $A_1 = B_1$, $A_2' = B_2'$ and $A_3'' = B_3''$ given in eq.(36), and therefore our result is consistent with that obtained by Klümper for the specific heat [10]. The specific heat per unit length of any model in the universality class including $S = 1/2$ Heisenberg antiferromagnetic chain obeys this result with $k = 1$. For example, the specific heat per unit length of $S = 1/2$ antiferromagnetic chain with a small perturbation of a second neighbor interaction satisfies the same formula by adjusting T_0. This shift occurs through the change of the initial coupling constant $g(a)$. As proposed in [3], this low temperature behavior of the specific heat per unit length (49) in the $S = 1/2$ ($k = 1$) Heisenberg chain will have to be observed experimentally, as well as its suceptibility. Eggert, Affleck and Takahashi calculated its suceptibility by one loop renormalization group and the Bethe ansatz for the Heisenberg chain [13]. The result has been observed by N. Motoyama, H. Eisaki and S. Uchida, experimentally in an antiferromagnetic chain Sr_2CuO_3 [5]. Here we discuss the data fitting of both the suceptibility and the specific heat which are observed. The Heisenberg hamiltonian is

$$H = J \sum_i {\rm S}_i \cdot {\rm S}_{i+1}.$$

The spin wave velocity is given by

$$v = \pi J/2,$$

which is determined from the Bethe ansatz. Eggert *et al.* gives the best fit $T_0 = 7.7J$ of the one loop renormalization group calculation for the suceptibility by the Bethe ansatz calculation of the Heisenberg chain [13]. They argued that the $\log^{-2} T_0/T$ term in the suceptibility can be eliminated by shifting T_0. In addition to the $\log^{-2} T_0/T$ term however, we have further logarithmic corrections from the two loop running coupling constant (34) in the one loop corrected suceptibility

$$\chi(T) = \frac{1}{2\pi v} + \frac{1}{4\pi v \log T_0/T}\left(1 - \frac{\log\log T_0/T}{2\log T_0/T}\right) + {\rm O}(\log^{-3} T_0/T).$$

This formula should be better for fitting the suceptibility calculated by the Bethe ansatz. To fit the specific heat at the same time, however, the adjusted T_0 for the suceptibility cannot be used for the specific heat with the same accuracy level as the suceptibility. This is because the same reson as in the fitting of numerical Bethe ansatz for the logarithmic finite size correction we argued in the previous section.

The elimination of both the $\log^{-2} T_0/T$ term in the suceptibility and $\log^{-4} T_0/T$ term in the specific heat cannot be done by shifting the only one parameter at the same time. We have to calculate the coefficient of $\log^{-2} T_0/T$ term in the suceptibility to fit both specific heat and the suceptibility. To do so, we have to calculate the g^2 order correction in the suceptibility explicitly.

6 Conclusion

In this paper, we have shown a simple calculation method for the higher order beta function of a perturbed CFT by utilizing the constraints given by Zamolodchikov's c-theorem. In a particular CFT model $\widehat{su(2)}_k \oplus \widehat{su(2)}_l/\widehat{su(2)}_{k+l}$, the known exact form of the OPE coefficient of the slightly relevant operator enables us to calculate the beta function up to three loop order in the ε-expansion, where $\varepsilon \equiv 4/(k+l+2)$. In the limit $\varepsilon \to 0$ as $l \to \infty$, we have the level k WZW model with a marginal perturbation. The obtained three loop beta function is useful to study a quantum spin chain model described in the marginally perturbed WZW model. The logarithmic finite size correction to the ground state energy in the $su(2)$ quantum spin chain model has been calculated up to the $\log^{-5} L$ order as well as the logarithmic temperature dependence of the specific heat. The obtained formula of finite size correction fits the data of a numerical Bethe ansatz quite well. The low temperature behavior of the specific heat obtained here should be observed experimentally. We provide a slightly better fitting function of the suceptibility than the one loop renormalization group calculation. Completely consistent treatment of both the suceptibility and the specific heat will be reported soon.

Acknowlegments

The author would like to thank M.-H. Kato for explaining his recent work and enlightening discussions. He thanks Y. Hatsugai for helpful comments on experimental physics. He is grateful to M. Itoi for helpful work of the data fitting.

Note added After writening this paper, the author discussed with N. Motoyama who explained that the spin specific heat of an antiferromagnetic chain Sr_2CuO_3 [5] cannot be separated from phonon specific heat with extremely large value compared to the spin part.

References

[1] A. B. Zamolodchikov, Sov. J. Nucl. Phys. **44** (1986) 529;
D. A. Kastor, E. J. Martinec and S. H. Shenker, Nucl. Phys. B**316** (1989) 590.

[2] A. B. Zamolodchikov, Sov. J. Nucl. Phys. **46** (1987) 1090;
A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. **B 285**(1987)687

[3] M. -H. Kato, "*Renormalization group flows and crossover scaling functions in deformed coset models*", Nihon university Ph.D. thesis, (1998).

[4] K. Nomura, Phys. Rev. B**48** (1993) 16814.

[5] N. Motoyama, H. Eisaki and S. Uchida, Phys.Rev.Lett. **76** (1996)1312.

[6] Al. B. Zamolodchikov, Nucl. Phys. B**366**(1991)122.

[7] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. B**346**(1990)409.

[8] Č. Crnković, G. M. Sotkov and M. Stanishkov, Phys. Lett. **B226** (1989) 297.

[9] M. Karbach and K.-H. Mütter, J. Phys. A **28** (1995) 4469.

[10] Klümper, "*The spin-1/2 Heisenberg chain: thermodynamics, quantum criticality and spin-Peierls exponents*", cond-mat/9803225.

[11] F. Woynarovich and H. -P. Eckel, J. Phys. A**20** (1987) L97 ; A**20**L443.

[12] I. Affleck, D. Gepner, H. J. Schulz and T. Zimann, J. Phys. A**22** (1989) 511.

[13] S. Eggert, I. Affleck and M. Takahashi, Phys. Rev. Lett. **73**(1994)332